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NON RANK 3 STRONGLY REGULAR GRAPHS WITH THE 5-VERTEX CONDITION

A. V. IVANOV

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Three new strongly regular graphs on 256, 120, and 135 vertices are described in this paper. They satisfy the t-vertex condition — in the sense of [1] — on the edges and on the nonedges for $t=4$ but they are not rank 3 graphs. The problem to search for any such graph was discussed on a folklore level several times and was fixed in [2]. Here the graph on 256 vertices satisfies even the 5-vertex condition, and has the graphs on 120 and 135 vertices as its subgraphs. The existence of these graphs was announced in [3] and [4]. [4] contains M. H. Klin's interpretation of the graph on 120 vertices. Further results concerning these graphs were obtained by A. E. Brouwer, cf. [5].

1. Introduction

The vertex set and edge set of an ordinary (undirected, without loops and multiple edges) graph G will be denoted by $V(G)$ and $E(G)$, respectively. We say that graph G_1 is a subgraph of graph G iff $V(G_1)\subset V(G)$ and for every $x, y\in V(G_1)$ $(x, y)\in E(G_1)\Leftrightarrow(x, y)\in E(G)$.

The following generalizes the definitions from [1].

Let G be an ordinary graph. We say that two subgraphs of G are of the same type with respect to the pair (x, y) of not necessarily distinct vertices $x, y\in V(G)$ if both contain x and y and there exists an isomorphism of one onto the other mapping x to x and y to y.

We say that the graph G satisfies the t-vertex condition on the edges (resp., on the nonedges or on the vertices), if for every i, $2\leqq i\leqq t$, the number of i-vertex subgraphs of every fixed type with respect to a pair (x, y) of adjacent (resp., nonadjacent or coinciding) vertices is the same for all edges (resp. nonedges or vertices).

If G satisfies the t-vertex conditions on the vertices, on the edges and on the nonedges then we say that G is a graph with the t-vertex condition. Obviously, a graph with the t-vertex condition is also a graph with the t'-vertex condition for every t': $2\leqq t'\leqq t$.

Examples of graphs with the t-vertex conditions are the regular graphs ($t=2$), strongly regular graphs ($t=3$), and rank 3 graphs $(2\leqq t\leqq|V(G)|)$.

It is clear that if the automorphism group Aut (G) of the graph G acts transitively on the vertices (resp., on the edges or on the nonedges) of G, then G satisfies the t-vertex condition $(2\leqq t\leqq|V(G)|)$ on the corresponding set.

One can show that a graph satisfying both the 2-vertex condition on the vertices and the t-vertex condition on the edges and on the nonedges satisfies the t-

AMS subject classification (1980): 05 C 25 vertex condition on the vertices as well (so, G is a graph with the t-vertex condition). The converse is not true in general. There are numerous well known examples of graphs with automorphism group transitive on the vertices which do not satisfy the 3-vertex condition.

It is easy to prove the following

Proposition 1. Let G be a graph with the 2-vertex condition having v vertices of valency k. If there are integers λ and μ such that for every distinct vertices $x, y \in V(G)$, $x \neq y$

$$\left|\{z \mid z \in V(G), (x, z), (y, z) \in E(G)\}\right| = \begin{cases} \lambda, & \text{if } (x, y) \in E(G) \\ \mu, & \text{otherwise} \end{cases}$$

then G is a graph with the 3-vertex condition and

$$k \cdot (k-\lambda-1) = (v-k-1) \cdot \mu.$$

It follows from this proposition that the class of graphs with the 3-vertex condition coincides with the class of strongly regular graphs, as stated above. Some other necessary conditions for the existence of strongly regular graphs can be found, for example, in [6].

In [1] the following is essentially proved.

Proposition 2. Let G be a graph with the 3-vertex condition having the parameters (v, k, λ). If there are integers α and β such that for every distinct vertices $x, y \in V(G)$, $x \neq y$

$$\left|\{(z^1, z^2) \mid (z^j, x), (z^j, y), (z^1, z^2) \in E(G),\ j = 1, 2\}\right| = \begin{cases} \alpha, & \text{if } (x, y) \in E(G) \\ \beta, & \text{otherwise} \end{cases}$$

then G is a graph with the 4-vertex condition and

$$k \cdot \left(\lambda \cdot (\lambda-1)/2 - \alpha\right) = (v-k-1) \cdot \beta.$$

Necessary and sufficient conditions which ensure that a graph with the t-vertex condition ($t \geq 4$) is a graph with the $(t+1)$-vertex condition as well seem to be more complicated.

It is known that a graph with the 3-vertex condition can have trivial automorphism group. On the other hand, all strongly regular graphs investigated until now by the author and his colleagues were graphs with the 4-vertex condition iff they were rank 3 graphs. So, the problem of finding for non rank 3 graphs with the 4-vertex condition [2, Problem 13] seems to be very natural.

A graph with the 5-vertex condition on 256 vertices and two of its subgraphs with the 4-vertex condition on 120 and 135 vertices, respectively, are described in the paper. None of these graphs are rank 3 and the automorphism group of the graph on 135 vertices acts intransitively on the vertex set.

We shall use the notion of cellular ring introduced in [7] (see also [2]). The symmetric binary relation on a set N and the corresponding basis graph of the cellular ring will be denoted in the same way.

Let A and B be symmetric binary relations on the set N. We shall denote by $A \oplus B$ a tensor product of the relations A and B. The relation $A \otimes B$ is defined on the Cartesian product $N \times N$ in the following way:

$$((x, y), (x', y')) \in A \otimes B \Leftrightarrow (x, x') \in A \quad \& \quad (y, y') \in B \quad \text{for} \quad ((x, y), (x', y')) \in N \times N.$$

We shall denote by $A \cup B$ the union of the relations A and B:

$$(x, y) \in A \cup B \Leftrightarrow (x, y) \in A \vee (x, y) \in B.$$

2. Graph on 256 vertices and its subgraphs

Using the method described in [8] the author has investigated a lattice of subrings of the symmetric squares of an amorphic cellular ring [9] of rank 4 [10]. (From the remark on p. 97 [6] one can suppose that an analogous approach was used by R. Mathon. Note that M. H. Klin has proposed the use of tensor products for the construction of strongly regular graphs as early as in 1974 [12].) In this way about 60 pairwise nonisomorphic graphs on 256 vertices of valencies 45, 60, 75, 90, 105, 120 were constructed. The graph presented below seems to be the one of greatest interest.

Let W be the cellular ring of rank 4 defined on the set N_W ($|N_W|=16$) with the subdegrees 1, 3, 6, 6, having automorphism group of order 192 isomorphic to $E_{16}\lambda D_6$ [10], and let I, A, B, C be its basis graphs (I is the reflexive graph, A and B are isomorphic to the lattice graph with $(v, k, \lambda)=(16, 6, 2)$ and C is the disjoint union of 4 copies of the complete graph K_4 on 4 vertices). Then the binary relation

$$A\otimes A\cup B\otimes B\cup A\otimes(I\cup C)\cup(I\cup C)\otimes A \tag{2.1}$$

defined on the set $N\times N$ gives a strongly regular graph G with the parameters $v=256$, $k=120$, $\lambda=\mu=56$. The automorphism group $\text{Aut}(G)$ of the graph G acts transitively both on the vertex set and on the edge set but intransitively on the set of nonedges of the graph. Under the action of the subgroup of $\text{Aut}(G)$ stabilizing the vertex $x\in V(G)$ the set of the 135 vertices nonadjacent with x is divided into two orbits of length 120 and 15. By direct counting of the representatives of two orbits of nonedges of the graph we get that $\beta_1=\beta_2=672$, where β_i, $i\in\{1, 2\}$ are the parameters defined in Proposition 2. Hence, in view of Proposition 2, we can maintain that the constructed graph is a graph with the 4-vertex condition. Direct calculations made by computer showed that the graph G is also a graph with the 5-vertex condition.

We analyzed some subgraphs of G. In what follows we shall denote by $G_1(x)$ (resp. by $G_2(x)$) the subgraph of the graph G induced by all vertices of G adjacent (resp. nonadjacent) to the vertex x. If the graphs $G_i(x)$ and $G_i(y)$, $i\in\{1, 2\}$ are isomorphic for any $x, y\in V(G)$ we shall write G_i instead of $G_i(x)$. We shall denote by $G_{i1}(y)$ (resp., by $G_{i2}(y)$) the subgraph of graph G_i induced by all vertices adjacent (resp., nonadjacent) to the vertex $y\in V(G_i)$.

Table 1 contains the parameters, the eigenvalues r and s, the values α and β and the orders of the automorphism groups of some subgraphs of the graph G. If a parameter can not be determined correctly for some graph we marked this fact by the sign "—". The order of the automorphism group $\text{Aut}(G_{21}(x))$ depends on x, as it is showed below.

Table 1

Graph	v	k	$v-k-1$	λ	μ	r	s	α	β	$\lvert\text{Aut}(G)\rvert$
G	256	120	135	56	56	8	-8	784	672	$2^{20}\cdot 3^2\cdot 5\cdot 7$
G_1	120	56	63	28	24	8	-4	216	144	$2^{12}\cdot 3^2\cdot 5\cdot 7$
G_2	135	64	70	28	32	4	-8	168	192	$2^{12}\cdot 3^2\cdot 5\cdot 7$
G_{11}	56	28	27	—	—	—	—	—	—	$2^{13}\cdot 3\cdot 7$
G_{12}	63	32	30	16	16	4	-4	—	—	$2^{9}\cdot 3\cdot 7$
$G_{21}(x)$	64	28	35	12	12	4	-4	—	—	—
$G_{22}(x)$	70	32	37	—	—	—	—	—	—	—

The graphs G, G_1 and G_2 are not rank 3 graphs.

The automorphism group $\mathrm{Aut}(G_1)$ acts transitively on the vertices and on the edges, but intransitively on the nonedges of the graph G_1. Under the action of the subgroup of $\mathrm{Aut}(G_1)$ stabilizing the vertex $x\in V(G_1)$ the set of 63 vertices nonadjacent to x is divided into two orbits of length 56 and 7. G_{11} is transitive and G_{12} is intransitive on the vertices.

The graph G_2 is the only known strongly regular graph with the 4-vertex condition which has an automorphism group intransitive on the vertex set.

If we choose a vertex $x\in V(G_2)$ from the orbit of length 120 then the set of vertices adjacent to x in the graph G_2 is divided into two orbits of length 56 and 8. The set of vertices nonadjacent to x is divided into three orbits of length 56, 7, 7 under the action of stabilizer of x from $\mathrm{Aut}(G_2(x))$. $G_{21}(x)$ is intransitive on the vertices and $|\mathrm{Aut}(G_{21}(x))|=2^9\cdot 3\cdot 7$.

If we choose a vertex $x\in V(G_2)$ from the orbit of lenth 15 then the vertices adjacent to x in the graph G_2 form one orbit of length 64. The set of vertices nonadjacent to x is divided into two orbits of length 56 and 14 under the action of stabilizer of x from $\mathrm{Aut}(G_2(x))$. $G_{21}(x)$ is isomorphic to the well known rank 3 graph on 64 vertices of valency 28 and $|\mathrm{Aut}(G_{21}(x))|=2^{13}\cdot 3^2\cdot 5\cdot 7$.

It was shown by direct examination that two symmetric (64, 28, 12)-designs, corresponding [17] to the graphs $G_{21}(x)$ and $G_{21}(y)$, where the vertices x, y are from two distinct orbits of $V(G_2)$, are not isomorphic. Only one such BIB-design was known before [13].

3. Interpretations of graphs G and G_1

Some attempts were made to describe the graph G and its subgraphs G_1 and G_2 in more convenient terms. They were successful for G and G_1.

The cellular ring W used by us for the construction of the graph G may be described in terms of Latin square of degree 4 corresponding to an elementary abelian group E_4 [10]. Using this description and (2.1) I. A. Faradjev constructed the following interpretation of G.

The vertex set of the graph G is the set of all different quadruples $\{(a, b, c, d)|a, b, c, d\in E_4\}$ of the elements from E_4. The vertices (a^1, b^1, c^1, d^1), (a^2, b^2, c^2, d^2) of G are joined by an edge iff

$$\neg(a^1 = a^2 \oplus b^1 = b^2 \oplus c^1 = c^2 \oplus d^1 = d^2) \quad \& \quad (a^1\cdot b^1 \neq a^2\cdot b^2 \vee c^1\cdot d^1 \neq c^2\cdot d^2)$$

is true. Here $\oplus$ is the logical sum modulo 2 and $\neg$ is the logical negation.

The following procedure from [11] was used for the interpretation of G.

Let $M=\|m_{ij}\|_{n\times n}$ be a Hadamard matrix. Take the set $V=\{(i,j)|\ 1\leq i,j\leq n\}$ of n^2 different elements of the matrix M as the vertex set of a new coloured graph $\Gamma=\Gamma(M)$ and colour the edge $\{(i,j),(s,t)\}$ with the colour k, where

$$k=0,\quad \text{if}\quad i=j,\quad s=t;$$

$$k=1,\quad \text{if}\quad i=j,\quad s\neq t;$$

$$k=2,\quad \text{if}\quad i\neq j,\quad s=t;$$

$$k=3,\quad \text{if}\quad i\neq j,\quad s\neq t,\quad m_{ij}\cdot m_{it}\cdot m_{sj}\cdot m_{st}=1;$$

$$k=4,\quad \text{if}\quad i\neq j,\quad s\neq t,\quad m_{ij}\cdot m_{it}\cdot m_{sj}\cdot m_{st}=-1.$$

As it was proved in [11] the described procedure gives an amorphic cellular ring of rank 5. So, by the definition of the amorphic cellular ring any partition of the set $\{R_i,\ 0\leq i\leq 4\}$ of the basis graphs corresponds to some cellular ring as well.

We start with the construction of the Hadamard matrix using Shrikhande's graph on 16 vertices (for general construction see, for example, [14]). Then we apply the procedure described above to this Hadamard matrix and take the partition of the set of non trivial basis graphs into two classes $R_1\cup R_3$ and $R_2\cup R_4$. As it was established by computer, the constructed graph is isomorphic to G.

For a description of an interpretation of G_1 proposed by M. H. Klin we need an auxiliary graph $\tilde{Q}_8$ and some well known properties of the distance transitive graphs [15], [16].

We start with the 8-dimensional cube Q_8 corresponding to the Hamming scheme $H(8,2)$. So, $Q_8=(V,E)$, where V is a vector space of dimension 8 over $GF(2)$. It is well known that $\mathrm{Aut}\,(Q_8)=S_2\dagger S_8\cong E_{256}\cdot S_8$.

Then we construct the graph $\bar{Q}_8$ with the vertex set V_0 two vertices of $\bar{Q}_8$ joined by an edge iff the distance between them in Q_8 is equal to 2. Considering that Q_8 is bipartite and $\mathrm{Aut}\,(Q_8)$ acts transitively on V, one can show that the graph $\bar{Q}_8$ consists of two isomorphic components. Denote anyone of them by $\hat{Q}_8$. It is easy to show that the vertex set of $V(\hat{Q}_8)$ consists of all vectors from V with an even number of coordinates equal to 1 (0) and $\mathrm{Aut}\,(\hat{Q}_8)=E_{128}\cdot S_8$.

$\hat{Q}_8$ is a halved graph of a distance-transitive graph and it is also a distance-transitive graph. Moreover, it is antipodal. Then we construct the graph $\tilde{Q}_8$ in the following way.

We take the set

$$\left\{\{(x_1^1,\ldots,x_8^1),(x_1^2,\ldots,x_8^2)\}\,\middle|\,(x_1^i,\ldots,x_8^i)\in V(\hat{Q}_8),\ x_i^1+x_i^2=1,\ 1\leq i\leq 8\right\}$$

as the vertex set of the graph $\tilde{Q}_8$. Two vertices $a,b\in V(\tilde{Q}_8)$,

$$a=\{(x_1^1,\ldots,x_8^1),(x_1^2,\ldots,x_8^2)\},\quad b=\{(y_1^1,\ldots,y_8^1),(y_1^2,\ldots,y_8^2)\}$$

are joined by an edge iff $\big(((x_1^i,\ldots,x_8^i),(y_1^j,\ldots,y_8^j))\big)$ is an edge of the graph $\hat{Q}_8$ for some $i,j\in\{1,2\}$. It is easy to show that $\mathrm{Aut}\,(\tilde{Q}_8)=\mathrm{Aut}\,(\hat{Q}_8)/Z_2\cong E_{64}\cdot S_8$. Note that the construction presented here is a particular case (for $n=8$) of the construction known under the name "folded halved n-cube".

One can show that $\tilde{Q}_8$ is isomorphic to Γ, where Γ is the graph on the vertex set F^6, $F=\{0,1\}$, and two vertices of Γ are adjacent iff their distance is 1, 2, 5 or 6. Γ is a strongly regular graph with $v=64$, $k=28$, $\lambda=\mu=12$. The automorphism group $\mathrm{Aut}\,(\Gamma)$ is a rank 3 group isomorphic to $E_{64}\lambda S_8$. $\mathrm{Aut}\,(\Gamma)$ acts transitively on the set M ($|M|=240$) of 8-vertex anticliques of Γ. This set is divided into two equivalent orbits under the action of the subgroup $E_{64}\lambda A_8$ of $\mathrm{Aut}\,(\Gamma)$ of index 2.

Let $\tilde{M}$ be one of these orbits. Let $\tilde{\Gamma}$ be the graph with the vertex set $\tilde{M}$. Two vertices of $\tilde{\Gamma}$ are adjacent iff the corresponding anticliques of Γ have exactly two common vertices. The automorphism group $\mathrm{Aut}\,(\tilde{\Gamma})$ is an imprimitive permutation group with subdegrees 1, 7, 56, 56. The isomorphism of G_1 and $\tilde{\Gamma}$ was established by direct calculation on the computer. Another interpretation of the graphs G and G_1 was described in [5].

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A. V. Ivanov

Institute for System Studies
9. Prospect 60 Let Oktyabrya
117312, Moscow, USSR